Rotation curves of spiral galaxies: A general-relativistic model

P. S. Negi [1]

Department of Physics, Kumaun University, Nainital-263 002, India

Abstract

Spiral galaxies are considered as static and spherically symmetric Dark Matter Configurations (DMC) [described by Tolman's type VII solution with vanishing surface density] of size $a \sim 10$ kpc in which $\sim 10^{11}$ Sun-like stars [non-zero rest-mass particles (NZRPs)] move along appropriate trajectories. Using general relativity (GR), we show that a mass of dark matter about 127 - 212 $\times 10^{11} M_\odot$ is required inside the sphere of size $a \sim 10$ kpc for agreement with the observed typical orbital velocity ranging from 150 to 250 km sec^{-1}. That is, on the average, the actual total mass accumulated in spiral galaxies is $M \cong (2.5 - 4.2) \times 10^{13} M_\odot$. Thus, even though the observed orbital velocities v satisfy the condition $(v \ll c)$, GR may have important consequences.

In our model, it is possible to obtain flat, slightly rising, and even declining rotation curves, which in fact represent the loci of various NZRPs originating simultaneously from near-central to outer regions of the galaxy. The model self-evidently explains the reason why most of the spirals show almost flat or slightly rising rotation curves and very few show declining curves. The NZRPs follow trajectories coming very close to the centre and are finally trapped in circles of minimum radii $\sim 0.03a - 0.005a$ with rotation velocities reaching $\sim 6544 - 8445$ km sec $^{-1}$. This is also true for NZRPs emitted in near-central regions, which get trapped even deeper. The velocity values are consistent with observations of central regions of various galaxies. This scenario may also lead to conditions suitable for evolution of relativistic supermassive stars and supermassive black holes at the centres of various galaxies.

Based on this study, we obtain the density parameter $\Omega \sim 0.127 - 0.212$, leading to an open model of the Universe with an age of $\sim 16.8 - 17.6$ Gyr [for the Hubble constant $H = 50$ km sec^{-1} Mpc^{-1}], which is significantly higher than the presently estimated age of globular clusters $\sim 13 - 15$ Gyr.

[1]e-mail: negi@upso.ernet.in

1. Introduction

For the recent three decades, a large amount of observational data on nearly flat and slightly rising rotation curves of spiral galaxies have pointed out that a lot of dark mass dominates in these galaxies (see, e.g., Rubin [1]). An explanation of these phenomena on the basis of the Newtonian gravitation theory (NGT) requires dark matter dominating in the halo region of various galaxies. But the amount of the dark matter which prevails among such galaxies cannot be made certain by using NGT-based models. However, some spiral galaxies with sharply declining rotation curves are also observed [2, 3]. Casertano and van Gorkom [3] interpreted the decrease in rotation velocity as an indication of a large ratio of luminous to dark matter in the luminous regions of these galaxies and thus weakening of the well-known "conspiracy" between luminous and dark matter [4, 5]. Thus, in order to explain simultaneously all three types of rotation curves, the models based upon NGT require individual assumptions like dominance of dark or luminous matter in the halo or in the luminous region of an individual galaxy, depending upon the nature of the rotation curve. So, no "common ground", *in general*, is available within NGT in explaining all three types of rotation curves observed in various spirals. The models based on the Modified Newtonian Theory (MONT) [6] (see, e.g., [7]) [of course, this theory requires no dark matter at all] would also require individual assumptions for individual galaxies, as NGT does, if the observational evidence of all three types of rotation curves is included.

Beside this problem, a number of observations regarding the central regions of various galaxies (including spiral galaxies) are now available, which indicate the existence of extremely high velocity ($\sim 10^4$ km sec $^{-1}$) of gas clouds and stars around the galactic centre [8–14]. The most widely believed explanation of this phenomenon is that supermassive black holes (SBHs) of masses $\sim 10^6 - 10^9 M_\odot$ are present at the centres of various galaxies. However, Maoz [15] emphasized that the presence of such SBHs will be proved only when various other options (such as, clusters of brown dwarfs, or low mass solar type stars) could be completely ruled out. The presence of such SBHs at the centres of various galaxies would also require a general-relativistic treatment, instead of the conventional Newtonian formalism. Based upon this argument and the belief that general relativity theory (GR) might explain phenomena at large scale, various authors have used GR to explain the nearly flat and slightly rising rotation curves of spiral galaxies [16–18]. Among various such models, the authors have generally considered static, spherically symmetric mass distribution, with a core specifying luminous matter and the halo governed by a $(1/r^2)$ density distribution [so that the mass in the halo region turns out to be proportional to the radial distance] with the equation of state $P = \gamma \rho$ (where P and ρ are the pressure and the energy-density, respectively, and γ is a constant). To terminate the $(1/r^2)$ density distribution at a finite radius (since this distribution in GR corresponds to a non-terminating solution), an envelope governing some other density distribution representing the luminous matter is matched to the galactic halo. Apparently, these models automatically give rising rotation curves in the halo region of the galaxies, which turn out to be nearly flat in the Newtonian limit (i.e., for the speed of light c approaching infinity), because the expression for the rotation velocity of a particle (i.e, star) is worked out by

assuming *arbitrarily* that the all particles move in circular orbits (see, e.g., [19]). The slope of these curves depends on the density distribution considered for the halo region, as is discussed in detail in Sec.3 of the present paper.

Thus, like NGT and MONT, various GR-based models as is discussed above, cannot provide a suitable explanation, *in general*, for all three types of rotation curves observed among various spiral galaxies. Furthermore, these models do not provide any reason(s) behind high-velocity gas clouds and stars observed around the central region of various galaxies.

Therefore, at present, the main problems associated with spiral galaxies can be summarized as follows: (i) to construct a suitable model of spiral galaxies which could possibly explain, in general, all three types of observed rotation curves, and (ii) to give a reasonable explanation to high-velocity gas clouds and stars observed near the central region of various galaxies, and (iii) if SBHs are, in fact, present at the centres of various galaxies, to trace out an actual scenario which could lead to their formation.

Taking all these points into account and realizing that GR is a theory of geodesics which represent the intrinsic properties of the space-time geometry, produced by a mass distribution, we take, as a basis of our study, some well-known observational data and avoid arbitrary assumptions, such as "all the particles move in circular orbits". We specify a spiral galaxy as a static, spherically symmetric dark matter configuration (DMC) in hydrostatic equilibrium, in which trapped luminous nonzero rest-mass particles (NZRPs), such as stars and gas clouds, move along their appropriate trajectories provided by the space-time curvature due to DMC. In fact, our basic assumption about the status of luminous matter in spiral galaxies as test-particles (unable to alter the space-time curvature of the whole galaxy specified as a DMC) turns out to be a consequence of the present study (this is what the GR demands) as is discussed later.

[Notice that here we are only proposing an alternative model of spiral galaxies based on broad observational evidences. We now do not consider, the problems related to protogalaxies, evolution of galaxies, etc. What we actually discuss is the location of NZRPs inside the galaxy, their initial velocities at this location, and their initial direction. These *initial conditions* of various NZRPs within the galaxy are worked out in such a manner that a general explanation of all three types of rotation curves, as well as the presence of high-velocity gas clouds near the central regions of various galaxies could be achieved.]

In NGT, the various galaxies are treated merely Newtonian because the mean velocity of constituent stars turns out to be around 300 km sec^{-1}, which corresponds to a dimensionless gravitational potential $(\phi/c^2) \sim 10^{-6}$ (equivalent to the "compaction parameter", to be discussed later) for the whole galaxy [20]. This result is based on considering a spiral galaxy as a system of self-gravitating particles [e.g., a normal spiral galaxy contains around 10^{11} Sun-like normal stars, each having a mass around $1M_{\odot}$ and a radius of the order of 10^6 km, in a sphere of radius around 10 kpc]. The gravitational force due to constituent stars and gas clouds is there balanced by the centrifugal force generated by the appropriate rotation of whole system around the galactic centre. In this manner, each

constituent of the galaxy can be regarded as moving in a circular orbit around the galactic centre, so that the locus of various particles constitutes a spiral-like curve. However, it should be noted that even for the planet Mercury, if we use GR, trajectories in the *exterior* field of the Sun (corresponding to the same value of dimensionless gravitational potential $\sim 10^{-6}$ on the surface) turn out to be very different from those obtained in NGT [though, one can use higher-order corrections in NGT (with respect to $(\phi/c^2) \sim 10^{-6}$ and higher) to retrieve the results of GR, but it would increase complications of the problem, rather than simplifying them]. Therefore, it may be argued that if we adopt a spiral galaxy as a DMC model and study the motion of NZRPs using GR, the results should be quite different from those of NGT or of the earlier GR-based models of spiral galaxies. The major results in this direction indicate that, apart from the nature of trajectories obtained for NZRPs, the DMC models could have the values of the surface dimensionless gravitational potential $\sim 10^{-4}$, that is, around two orders of magnitude higher than that obtained by using the NGT.

Instead of assuming a spiral galaxy as a system of self-gravitating particles, we describe it as a DMC model with a radius of the order of 10 kpc [such a region contains nearly all luminous matter present in spiral galaxies], in which around 10^{11} Sun-like stars move along trajectories which can be worked out by imposing suitable initial conditions. Note that, in this model, all constituent stars do not necessarily move in circular orbits at each radial distance from the centre, as is necessary in the Newtonian formulation. The present type of model is also inspired by the results of various cosmological studies which indicate that more than 90% of the mass of the Universe is contained in the dark component of unknown nature; it interacts with the luminous matter only through the gravitational interaction. We wish to accumulate this large amount of dark matter inside spiral galaxies as DMC models considered in the present study. As in the Newtonian case, the locus of various NZRPs situated from near the centre to the surface of the galaxy, comprise a spiral-like curve, but their orbital velocities are regarded as part of the *initial conditions*. By a suitable choice of these "initial conditions", we can obtain models of spiral galaxies corresponding to nearly flat, slightly rising, or even declining rotation curves.

For example, we can use the observed mean velocity of constituent stars about 300 km sec^{-1} as an *initial* velocity of a star located at a particular position (radial distance from the centre) with proper direction. Apparently, the dimensionless gravitational potential of the DMC will not necessarily correspond to a value $\sim 10^{-6}$ as in the Newtonian case. Instead, we have to work out its value by imposing some other observational constraint regarding these galaxies. Such a constraint could follow, e.g., from the typical values of observed rotation velocities in various types of spiral galaxies, corresponding to the range of 150 to 250 km sec $^{-1}$ from type Sc to type Sa, respectively. Imposing these constraints and using the general expression for orbital velocity in GR, we obtain the compaction parameter, equivalent to the dimensionless gravitational potential: $u[\equiv$ total mass to size ratio, (M/a), in geometrized units (see the Appendix for definitions). The range of u is $u \simeq (1.27 - 2.12) \times 10^{-4}$). Note that the gravitational potential obtained in this manner using GR turns out to be about two orders of magnitude greater than the presently believed value and cannot be obtained using NGT or MONT. Thus, it is apparent that even when the ratio, (v^2/c^2) [v is the test-particle velocity and c the speed

4

of light in vacuum] is small compared to unity, GR may have important consequences, in particular for *test-particle motion* (rather than *merely structural properties*) in the *interior* field of DMC-like structures, as models of real galaxies.

It follows that spiral galaxies as DMC-like structures may contain a dark mass which is 127 - 212 times higher than the luminous mass (observed in the form of stars and gas clouds in the spiral arms), independently of the nature of the rotation curve, be it flat, slightly rising or even declining. (The latter is unlikely in the Newtonian theory, in which, to explain a sharply declining rotation curve, a spiral galaxy must have luminous matter dominance in the luminous region.) Thus the DMC model considered in the present study not only justifies our basic assumption about spiral galaxies (that the luminous matter can be treated as test-particles moving inside the DMC), but also provides a suitable general logical explanation of all observed types of rotation curves, which was lacking in Newtonian models.

Another interesting feature of this study, which can not be obtained in NGT, is that various trapped NZRPs form a spiral-like curve, corresponding to the trajectories (shown in Fig. 2 - 4 by dashed lines) approaching very close to the centre and then trapped always into a circular orbit of minimum radius r_{min} given by Eq.(27), that is, circular orbits in GR-models are possible only at a minimal radius, as opposed to any radius in NGT (in which the motion of every constituent of the galaxy in a circular orbit is necessary to maintain the equilibrium). The final orbital velocities of such trapped NZRPs (in circular orbits) reach $\sim 10^4$ km sec^{-1}, which is consistent with the observations regarding the central regions of various galaxies. Thus the DMC models of spiral galaxies studied here may provide a suitable alternative to the belief that there are SBHs at the centres of various galaxies. The point is that, if the high velocity of gas clouds and stars are explained by the presence of SBHs, the rotation curves near galactic centres should necessarily be Keplerian in nature; however, this feature is not supported by recent observations of central rotation curves of spiral galaxies which generally show a steep nuclear rise and a high rotation velocity, common to all massive galaxies [21]. If, however, the future observations prove the existence of SBHs at the centres of various galaxies, the present scenario can provide conditions, suitable for evolution of relativistic supermassive stars and SBHs [22]. Though, we do not want to stress this point any more; this scenario could provide an appropriate basis for future studies.

In Sec.2, we obtain the necessary condition for trapping of NZRPs at various points inside the DMC. For this purpose, we obtain the trapping angle ψ_0, that is, the maximum semi-angle of the "cone of avoidance" (an emitted NZRP exceeding ψ_0 is trapped by the DMC).

In Sec.3, we obtain an expression for the orbital velocity of an NZRP from the equations of motion in GR. Variation of the NZRP orbital velocity is found to be inversely proportional to the radial distance r. In other words, for a given NZRP, one always obtains a declining rotation curve from near the centre to the outer region of the configuration. However, it should be noted that measurement along such a curve for an assigned NZRP on the scale of a galaxy has no practical significance since it would require a period of the order of megayears. Then, what can be the reason for the large number of observa-

tional data on flat and slightly rising rotation curves in most of the spiral galaxies? This question can be well answered if we assume that we are at present, in fact, *watching the locus of various trapped NZRPs inside the DMC, which comprise the spiral-like curve, and measuring their initial orbital velocities at the moment of their origin.*

To understand this scenario more clearly, we show in the next section that test-particle trapping depends only on the *initial conditions*, including the initial orbital velocity. And these initial conditions, in fact, allow us to obtain flat, rising, or even declining orbital velocities of various NZRPs, originating simultaneously from various points inside the DMC. The *loci* of these NZRPs can be easily arranged according to the observational situation, e.g., any specific shape of the spiral. Furthermore, we also obtain the corresponding NZRP trajectories for various initial conditions adopted in this paper. As is discussed later, the *initial* orbital velocities at various points inside the structure depend mainly on the emission angle ψ_i, which provides the reason why the majority of spirals show almost flat or slightly rising rotation curves.

Furthermore, it is well known that the population of the Universe is dominated by spiral galaxies. The density parameter $\Omega \equiv \rho_0/\rho_{crit}$ (the ratio of present mass density to the critical density of the Universe) $\sim 0.002 - 0.003$ for galaxies alone, which includes a dark mass of 3 to 5 times the mass of luminous matter [1, 23, 24]. However, if we assume that most of the dark matter in the Universe is more or less distributed among galaxies (in a manner in which spiral galaxies may contain a dark mass of 127 - 212 times the mass of luminous matter), then the density parameter will be at most $\Omega \sim 0.127$ - 0.212, leading to an open model of the Universe with an age estimate of about 16.8 - 17.6 Gyr (for the Hubble constant $H = 50 \, \text{km} \, \text{sec}^{-1} \, \text{Mpc}^{-1}$ as used by Rubin [1] for observations of spiral galaxies). This age estimate is significantly higher than the age of globular clusters $\sim 13 - 15$ Gyr [25, 26].

2. Equations of motion and test-particles trapping in static, spherically symmetric mass distributions

The metric corresponding to a static, spherically symmetric mass distribution can be written as

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \tag{1}$$

where $G = c = 1$ and ν and λ are functions of r alone. The resulting field equations for systems with isotropic pressure P and energy-density ρ are well known and comprise the standard textbook material.

Defining an angle ψ [as shown in Fig.1, in which we use, for convenience, the radial distance $y(\equiv r/a)$ in units of configuration size] as the angle between the radial direction and tangent to the orbit $r = r(\phi)$ in the $\theta = (\pi/2)$ plane, we can write

$$\tan \psi = re^{-\lambda/2}(d\phi/dr). \tag{2}$$

Let J be the specific angular momentum [(angular momentum/rest-mass)] and E be the specific energy [(energy/rest-mass)] of a test-particle in this plane, so that the ratio (J/E)

represents the "impact parameter" and is a constant of motion [20, 27, 28], that is

$$b = (J/E). \tag{3}$$

For the metric (1), the equations of motion are [29]

$$\theta = \text{constant} = (\pi/2) \text{ say}, \tag{4}$$
$$r^2(d\phi/ds) = J = \text{constant}, \tag{5}$$
$$e^\nu(dt/ds) = E = \text{constant}, \tag{6}$$
$$e^\lambda(dr/ds)^2 = e^{-\nu}(E^2 - e^\nu[(J/r)^2 + 1]). \tag{7}$$

Eqs.(3), (4), and (7) indicate that the orbit is confined to a plane (for convenience, we take it as the equatorial plane, $\theta = \pi/2$). In the metric (1), we get

$$e^\lambda(dr/ds)^2 = e^\nu(dt/ds)^2 - r^2(d\phi/ds)^2 - 1.$$

Applying the conditions (5) and (6), we obtain

$$e^\lambda(dr/d\phi)^2 = e^{-\nu}(E^2/J^2)r^4 - (r^4/J^2) - r^2$$

and finally, using Eq.(3) for the impact parameter, we get

$$e^\lambda(dr/d\phi)^2 = r^2 e^{-\nu}(r/b)^2 - r^2 E^{-2}(r/b)^2 - r^2, \tag{8}$$

or,

$$(d\phi/dr) = (b/r)e^{-\lambda/2}/[r^2(e^{-\nu} - E^{-2}) - b^2]^{1/2}, \tag{9}$$

which is the required equation describing the test-particle trajectory [an NZRP or ZRP (zero rest-mass)]. The difference in the ZRP and NZRP trajectories follows from the term $(r^2 E^{-2})$ in the denominator of the right hand side of Eq.(9) [30].

From Eqs.(2) and (8), we get

$$\sin\psi = (b/r)(e^{-\nu} - E^{-2})^{-1/2}, \tag{10}$$

when $\psi = 90°$,

$$b = b(90°) = r(e^{-\nu} - E^{-2})^{1/2}, \tag{11}$$

and $b(90°)$ attains its minimum value, $b_{min}(90°) = ß$, when

$$(d\nu/dr) = (2/r)(1 - e^\nu E^{-2}). \tag{12}$$

Now, defining an angle ψ_0 given by

$$\sin\psi_0 = (ß/r)(e^{-\nu} - E^{-2})^{-1/2}, \tag{13}$$

whose physical importance lies in the fact the when $\psi > \psi_0$, the emitted test-particles will be gravitationally trapped inside the spherical configuration. Thus ψ_0 represents the maximum semi-angle of the cone at a radial distance r such that only particles emitted inside this cone will be able to escape the mass distribution.

7

Let $u[\equiv (M/a)$, total mass to size ratio] represents the compaction parameter of the spherical configuration, and, for convenience, we will measure the impact parameter b and the radial coordinate r in units of the configuration size a, so that

$$\beta \equiv (b/a), \text{and} \tag{14}$$

$$y \equiv (r/a). \tag{15}$$

In terms of these variables, Eqs.(9), (10), and (13) become

$$(d\phi/dy) = (\beta/y)e^{-\lambda/2}/[y^2(e^{-\nu} - E^{-2}) - \beta^2]^{1/2}, \tag{16}$$

$$\beta = y(e^{-\nu} - E^{-2})^{1/2} \sin \psi, \tag{17}$$

$$\sin \psi_0 = (2u\beta/y)(e^{-\nu} - E^{-2})^{-1/2}. \tag{18}$$

3.   General expression for the NZRP orbital velocity inside a DMC

The orbital velocity v_ϕ of a NZRP moving inside the DMC governed by the metric (1) is

$$v_\phi = r(d\phi/dt), \tag{19}$$

which can be written as

$$v_\phi = r(d\phi/ds)(dt/ds). \tag{20}$$

Substitution of Eqs.(5) and (6) into Eq.(20) gives

$$v_\phi = r(J/r^2)(e^\nu/E). \tag{21}$$

Finally, using Eq.(3) for the impact parameter b in Eq.(21), we get

$$v_\phi = (b/r)e^\nu \tag{22}$$

(Apparently, a constant orbital velocity v_ϕ of any given NZRP with the impact parameter b is obtained for a structure with

$$e^\nu \propto r,$$

or,

$$e^\nu = Cr, \qquad C = \text{const.} \tag{23}$$

It is interesting to note that a configuration of this type corresponds to the well-known non-terminating exact solution with the density variation $\rho \propto (1/r^2)$ and the pressure $P = (\rho/3)$, which lead to unphysical conditions in the sense that both the pressure and density are infinite at the centre.)

One can mention that Hojman et al. [19] have obtained an expression for the orbital velocity, $v_{\phi,c}$ (say) by assuming that all test-particles move in circular orbits, that is, they have simply substituted the value of $(d\phi/dt)$ under this *arbitrary* assumption into Eq.(19) (see, e.g., Eq.(4.8.25) on p. 188 of Ref. [31]), and obtained the orbital velocity

$$v_{\phi,c} = [(r/2)d(e^\nu)/dr]^{1/2}. \tag{24}$$

Using the TOV [32, 33] equation, Eq.(24) can be written as

$$v_{\phi,c} = (e^{(\nu+\lambda)/2}/\sqrt{2})(8\pi P r^2 + 1 - e^{-\lambda})^{1/2}. \tag{25}$$

Eq.(25) indicates that for any regular (in the sense discussed in [34]) density distribution, one always obtains a rising rotation curve (since both e^{ν} and e^{λ} are increasing functions of r in any regular solution). The slope of the curves are determined by the density distribution considered to specify the galaxy. One may choose the density variation required according to the problem. For example, one may choose between fastest variation of density ($\propto 1/r^2$) to obtain almost constant rotation curves and the smoothest density variation (i.e., the constant density solution) to obtain the steepest rising rotation curve. However, such models are unable to give declining rotation curves. Furthermore, the total amount of dark matter in such models is not certain. Moreover, if one wishes to obtain almost flat rotation curves by using the $1/r^2$ density distribution (as used in [19]), it makes the model artificial since, to avoid the central singularity of the $(1/r^2)$ density distribution, one has to replace the central core of the model with some other, nonsingular density distribution.

We can write Eq.(22) in terms of the variables β and y [Eqs.(14) and (15)] as

$$v_{\phi} = (\beta/y)e^{\nu}. \tag{26}$$

The mean stellar velocity in various galaxies corresponds to a value of ~ 300 km sec^{-1} [20], which gives the specific energy of a star $E \cong 1.0000$. If we substitute the typical value of the orbital velocity $\sim 150 - 250$ km sec^{-1}, measured in various spiral galaxies (see, e.g., Fig. 2 of [1]) into Eq.(26), we obtain the typical value of the compaction parameter u of these galaxies as $u \sim (1.270 - 2.115) \times 10^{-4}$ [e.g., substituting (β/y) from Eq.(17) into Eq.(26) for $\psi = \psi_i > \psi_0$ and $E = 1$, as shown in Table 3, and using the boundary condition that at $r = a$ the internal solution matches with the exterior Schwarzschild solution, that is, at $y = 1$, $e^{\nu} = e^{-\lambda} = (1 - 2u)$]. This is about 127-212 times higher than the typical value of the compaction parameter, $u \sim 10^{-6}$, presently believed for these galaxies.

Notice that for the value of $u(\sim 10^{-4})$ obtained here, the metric function e^{ν} remains almost constant ($\cong 0.9999$) from centre to the outer region of the configuration, and Eq.(26) always corresponds to a declining rotation curve for *a given NZRP*. However, as we have discussed in detail in Sec.1, for a typical spiral galaxy (size $\sim 10^4$ pc), the time scale required to measure the deviation in the orbital velocity for a *particular* NZRP is of the order of megayears, which is meaningless in practice (even if the measurement could be possible). Therefore, the only remaining physically viable option (which also seems to be likely) to explain various observations of flat, slightly rising, and even declining orbital velocities in spiral galaxies is to assume that we are observing the *loci* of various trapped NZRPs, originated simultaneously from various points inside the structure from central to the outer region of the galaxy. According to Eq.(26), the initial orbital velocity of a NZRP, originating from a specific point (y_i), depends mainly on b (which is a constant of motion and depends only on the *initial conditions*, see. e. g. [28]). Thus we can construct various models of spiral galaxies, corresponding to flat, slightly rising, and even declining (initial)

9

orbital velocities of its constituents (NZRPs), by reconstruction of the *initial conditions* of the NZRPs. The *loci* of these NZRPs give specific shapes of the spirals, which may be obtained according to the choice of the *initial conditions.* A possible explanation of the observational fact that the majority of spiral galaxies show almost flat or slightly rising rotation curves is given in the next section.

4. Discussion: constant, slightly rising, and declining rotation curves of spiral galaxies. The DMC model and the value of Ω

To discuss the proposed models, we first work out the term $(\text{ß}/y)$ appearing in Eq.(18) and then find the trapping angles ψ_0 for various NZRPs ($E \cong 1.0000$) emitted from various points ($y = y_i$) inside a static and physically realistic mass distribution described by Tolman's type VII solution with vanishing surface density [32, 35, 36, 22]. Table 1 shows various $\text{ß}/y_i$ values at different points ($y = y_i$) within the structure. Using Table 1, we have obtained the trapping angles ψ_0 as shown in Table 2 for various y values inside the configuration corresponding to some typical values of u. The solution is given in a simple and convenient form in [35, 36, 22]. [Note that one may use some other physically realistic equation of state or exact solution for DMC, but the compaction parameter of the whole configuration will remain the same, keeping the results unaffected. However, in those cases the calculations for obtaining the trapping angles would be rather complicated.]

Having obtained the trapping angles ψ_0, we can work out constant, slightly rising and even declining instantaneous orbital velocities corresponding to various trapped NZRPs (with $\psi > \psi_0$) emitted simultaneously from near-central to outer region of the configuration, such that their *loci* comprise a spiral-like curve with a specific shape according to the initial conditions. Table 3 describes various *initial conditions* imposed on NZRPs emitted from the near-central region (e.g., $y = 0.2$) to the surface ($y = 1.0$) of the configuration. Their initial orbital velocities are (i) almost constant, (ii) slightly rising, and (iii) even declining throughout the configuration ranging from 100 to 339 km sec^{-1}, corresponding to configurations with compaction parameters 1.270×10^{-4}, 1.710×10^{-4}, and 2.115×10^{-4}, respectively.

For illustration, we have drawn the *loci* of NZRPs for the initial conditions corresponding to the first three columns of Table 3. The loci indicated by the points A, B, C, D, E form a spiral-like curve and correspond to an almost constant orbital velocity, $v_\phi \cong 203$ km sec^{-1} [Fig.2]. The compaction parameter of the structure turns out to be $\sim 1.71 \times 10^{-4}$. Similarly, as shown in Fig.3, the NZRPs indicated by the points A, B, C, D, E, are emitted from the points, $y = 1.0, 0.8, 0.6, 0.4,$ and 0.2 inside the configuration with the compaction parameter, $u \sim 2.12 \times 10^{-4}$. They comprise a slightly rising rotation curve, with orbital velocities ranging from 200 to 251 km sec^{-1}. The locus of the NZRPs marked with A, B, C, D, E in Fig.4 corresponds to a declining rotation curve . The orbital velocity decreases from point E (~ 203 km sec^{-1}) to point A (~ 150 km sec^{-1}) inside a configuration with the compaction parameter $u \sim 1.27 \times 10^{-4}$.

The actual trajectories of these NZRPs (A, B, C, D, E) are also obtained using Eqs.(16) and (17) for the metric parameters e^ν and e^λ given by Tolman's type VII solution with vanishing surface density [35, 36, 22] and shown in the respective figures (2-4) with dashed

lines. In all trajectories a test-particle moves inward until it reaches a minimum distance (y_{min}) given by the equation [28, 35]

$$y^2_{min}[(e^{-\nu})_{min} - E^{-2}] - \beta^2 = 0, \tag{27}$$

where $(e^{-\nu})_{min}$ is the value of $e^{-\nu}$ at $y = y_{min}$, and $\beta = y_i((e^{-\nu})_i - E^{-2})^{1/2} \sin \psi_i$. Then it remains confined in a circular orbit at the minimum distance (y_{min}).

Table 4 indicates various values of this minimum distance (y_{min}) from the centre of each configuration considered. The small circles in Figs.2-4 indicate the boundary of the central region where all NZRPs considered here are trapped in circular orbits of the radius y_{min}. The final orbital velocity at $y = y_{min}$ is from 6544 to 8445 km sec^{-1} [note that this velocity range also corresponds to various NZRPs trapped within $y = 0.05$ (as shown in Figs. 2-4), even if they are emitted from inside this circle, because they are always trapped after following the corresponding trajectories ending with a circular orbit given by Eq.(27).]. The values of escape velocities at the minimum distance y_{min} [denoted $v_{esc}(y_{min})$] and at the surface of the structure [denoted $v_{esc}(a)$] are also shown in Table 4. It is apparent that the values of escape velocities are always higher than the particle velocities at the respective points. The findings of this paper might be taken as a successful alternative to the present assumption on the presence of SBHs at the centres of various galaxies (made to explain various observations of high gas velocities near galactic centres). However, if SBHs are certainly present at the centres of some or many galaxies, the present scenario can provide a clue for explaining their formation (see also [35, 22]).

Now, we turn our attention to the observational evidence that most of the spirals show almost flat or slightly rising rotation curves and its possible explanation based on the present model. Consider, e.g., the trapping angle ψ_0 at the surface of the configuration $(y = 1.0)$ with the compaction parameter $u = 1.71 \times 10^{-4}$, corresponding to the value $177°.882$ [Table 2]. The NZRP emitted from any point inside the configuration at an angle $\psi_i > 177°.882$ will be gravitationally trapped. As shown in Table 3, NZRPs emitted from any point inside the structure $(u = 1.71 \times 10^{-4})$ at an angle $\psi_i = 177°.906$, constitute a declining rotational curve from the central region to the surface, whereas those emitted at an angle $\psi_i > 177°.906$ constitute a slightly rising or almost flat rotation curve. Evidently, the possibility of NZRP emission at an angle $\psi_i > 177°.906$ is always higher than the same near $\psi_i = 177°.906$, probably explaining the reason of why the majority of spiral galaxies show almost flat or slightly rising rotation curves.

Furthermore, we can estimate the density parameter $\Omega[\equiv (\rho_0/\rho_{crit})$, where ρ_0 is the present mass density of the Universe] in the following manner. It is well known that the population of the Universe is dominated by spiral galaxies. The density parameter Ω for these galaxies alone (including dark matter of 3 to 5 times the luminous matter density in a visible galaxy, with a typical size ~ 10 kpc) is estimated as $\sim 0.002 - 0.003$ [1, 23, 24]. According to the present study, we estimate that a typical galaxy contains a dark matter mass about 127 to 212 as much as luminous matter. If we assume that our description applies to all spiral galaxies and, as was mentioned above, that the Universe is dominate by these galaxies, the density parameter of the Universe will be at most about 0.127-0.212, leading to an open model of the Universe with an age estimate around 16.8-17.6 Gyr (for

the Hubble constant $H = 50$ km sec^{-1} Mpc^{-1} as regarded by Rubin [1] for observations of spiral galaxies). This is significantly higher than the globular clusters age $\sim 13 - 15$ Gyr [25, 26].

Acknowledgment

Appendix

The compaction parameter u is a dimensionless parameter which represents the total mass to size (radius) ratio of a static spherical configuration. Thus it is a measure of its compactness, saying how much mass is accumulated in a sphere of radius a so that the configuration does not lose its hydrostatic equilibrium. The parameter is defined as

$$u = GM/ac^2, \tag{28}$$

where M is the total mass and a is the radius of the configuration; c is the speed of light in vacuum, and G is the Newtonian gravitational constant.

GR is essentially a theory of gravity in which the gravitational force is regarded as an in built property of the space-time curvature produced by a mass distribution. Therefore, it is convenient to measure all physical quantities (like mass, density, pressure, etc.) in units of length. This can be done by choosing

$$G = c = 1, \tag{29}$$

so that we obtain various quantities in "geometrized units" as: $u \equiv (M/a)$, $1\,\mathrm{gm} \equiv 0.742 \times 10^{-28}\,\mathrm{cm}$, $1\,\mathrm{sec} \equiv 3 \times 10^{10}\,\mathrm{cm}$, etc. Thus, in particular

$$1 M_\odot = 2 \times 10^{33}\,\mathrm{gm} \equiv 1.484\,\mathrm{km} \equiv 5 \times 10^{-16}\,\mathrm{kpc}. \tag{30}$$

It is apparent from Eq.(28) that the "compaction parameter" is equivalent to the "dimensionless gravitational potential" (i.e., the gravitational potential in units of c^2), which is the "dimensionless gravitational energy per unit mass". However, an exact expression for the gravitational energy per unit mass in GR, denoted by α_p, in terms of u is given by [20]

$$\alpha_p = (u_p/u) - 1, \quad u_p = (M_p/a), \tag{31}$$

where a is the radius and M_p is the proper mass of the configuration given by

$$M_p = \int_0^a 4\pi \rho r^2 e^{\lambda/2} dr \tag{32}$$

Here ρ is the energy-density and e^λ is the metric coefficient such that

$$e^{-\lambda} = 1 - [2m(r)/r], \tag{33}$$

$m(r)$ being the mass contained inside the radius r.

13

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Table 1.

$E = 1.0000$		
y_i	u_i	$\text{ß}/y_i$
0.2000	0.1630	3.3955
0.4000	0.1696	3.1909
0.6000	0.1827	2.8633
0.8000	0.2071	2.4413
1.0000	0.2500	2.0000

Table 2.

$u(10^{-4})$	$E = 1.0000$				
	ψ_0 for various y_i				
	0.2000	0.4000	0.6000	0.8000	1.0000
$\downarrow$					
1.27000	177°.706	177.756	177.847	177.990	178.174
1.71000	177.338	177.396	177.502	177.667	177.882
2.11500	177.040	177.104	177.221	177.406	177.644

Fig.1: The azimuthal angle ϕ and the emittance angle ψ, at a radial distance $y(\equiv r/a)$ from the centre of spherical configuration in $\theta = (\pi/2)$ plane. 0 and a are the centre and radius of the spherical configuration, respectively. A test-particle is initially emitted from a point at radial distance $y = y_i$ with $\psi = \psi_i$ and $\phi = \phi_i$.

Table 3.

$u(10^{-4})$	y_i	ψ_i	$v_\phi(\mathrm{km\,sec^{-1}})$	ψ_i	$v_\phi(\mathrm{km\,sec^{-1}})$	ψ_i	$v_\phi(\mathrm{km\,sec^{-1}})$
			$E = 1.0000$				
1.710	0.2000	178°.450	202.8	178°.852	150.2	177°.906	273.9
1.710	0.4000	178°.387	202.8	178°.720	160.9	177°.906	263.1
1.710	0.6000	178°.275	202.8	178°.506	175.6	177°.906	246.1
1.710	0.8000	178°.111	202.8	178°.275	185.2	177°.906	224.8
1.710	1.0000	177°.906	202.8	177°.906	202.8	177°.906	202.8
2.115	0.2000	178°.276	250.8	178°.624	200.2	177°.670	339.0
2.115	0.4000	178°.205	250.9	178°.496	210.2	177°.670	325.6
2.115	0.6000	178°.081	250.8	178°.272	225.9	177°.670	304.5
2.115	0.8000	177°.898	250.9	178°.018	236.6	177°.670	278.1
2.115	1.0000	177°.670	250.9	177°.670	250.9	177°.670	250.9
1.270	0.2000	178°.669	150.1	179°.109	100.5	178°.203	202.6
1.270	0.4000	178°.615	150.0	178°.932	115.7	178°.203	194.6
1.270	0.6000	178°.519	150.0	178°.758	125.8	178°.203	182.0
1.270	0.8000	178°.378	150.1	178°.532	135.8	178°.203	166.2
1.270	1.0000	178°.203	150.0	178°.203	150.0	178°.203	150.0

Table 4.

$u(10^{-4})$	y_i	$\beta(10^{-4})$	y_{min}	$v_\phi(y_{min})(\mathrm{km\,sec^{-1}})$	$v_{esc}\,(a)(\mathrm{km\,sec^{-1}})$	$v_{esc}(y_{min})\,(\mathrm{km\,sec^{-1}})$
			$E = 1.0000$			
1.710	1.0	6.7594	0.0267	7592.0	5548.1	7595.3
1.710	0.8	5.4080	0.0214	7592.7	5548.1	7596.0
1.710	0.6	4.0563	0.0160	7593.2	5548.1	7596.7
1.710	0.4	2.7035	0.0107	7593.5	5548.1	7596.7
1.710	0.2	1.3523	0.0053	7593.6	5548.1	7596.7
2.115	1.0	8.3645	0.0297	8442.0	6170.6	8446.1
2.115	0.8	6.3101	0.0224	8444.0	6170.6	8448.0
2.115	0.6	4.5187	0.0160	8444.7	6170.6	8448.6
2.115	0.4	2.8039	0.0099	8444.0	6170.6	8448.6
2.115	0.2	1.3353	0.0047	8444.6	6170.6	8449.2
1.270	1.0	4.9999	0.0229	6543.8	4781.5	6546.1
1.270	0.8	4.4333	0.0203	6543.7	4781.5	6546.1
1.270	0.6	3.6412	0.0167	6544.7	4781.5	6546.9
1.270	0.4	2.5956	0.0119	6544.7	4781.5	6546.9
1.270	0.2	1.3509	0.0062	6545.2	4781.5	6547.7

Fig.2: The spiral-like curve is the *locus* of NZRPs emitted initially from the points with the following coordinates $(y_i,\ \psi_i,\ \phi_i)$: A (1.0, 177°.906, 0°), B (0.8, 178°.111, 10°.0), C (0.6, 178°.275, 30°.0), D (0.4, 178°.387, 50°.0), E (0.2, 178°.450, 60°.0), such that their (initial) orbital velocity remain constant, about 203 km sec^{-1}. The dashed curves represent the NZRP trajectories ending in circular orbits of minimum distances y_{min} with finally reached values of the orbital velocity $v_\phi(y_{min}) \sim 10^4$ km sec^{-1}, as shown in Table 4. These NZRPs are trapped in circular orbits of radius y_{min} in the region shown by a small circle of radius $y = 0.05$. The compaction parameter is $u = 1.71 \times 10^{-4}$.

Fig.3: The spiral-like curve is the *locus* of NZRPs emitted initially from the points with the following coordinates $(y_i, \ \psi_i, \ \phi_i)$: A (1.0, 177°.670, 0°), B (0.8, 178°.018, 30°.0), C (0.6, 178°.272, 70°.0), D (0.4, 178°.496, 100°.0), E (0.2, 178°.624, 150°.0), such that their (initial) orbital velocities slightly rise from 200 km sec^{-1} at point E to $\sim$ 251 km sec^{-1} at point A. The dashed curves represent the NZRP trajectories ending in circular orbits of minimum distance y_{min} with finally reached orbital velocities $v_\phi(y_{min}) \sim$ 10^4 km sec^{-1}, as shown in Table 4. The boundary inside which these NZRPs are trapped in circular orbits of radius y_{min} is shown by a small circle of radius $y = 0.05$. The compaction parameter is $u = 2.115 \times 10^{-4}$.

Fig.4: The spiral-like curve is the *locus* of NZRPs emitted initially from the points with the following coordinates $(y_i, \ \psi_i, \ \phi_i)$: A (1.0, 178°.203, 0°), B (0.8, 178°.203, 20°.0), C (0.6, 178°.203, 40°.0), D (0.4, 178°.203, 70°.0), E (0.2, 178°.203, 110°.0), such that their (initial) orbital velocities decline from 202.6 km sec^{-1} at point E to about 150 km sec^{-1} at point A. The dashed curves represent the NZRP trajectories ending in circular orbits of minimum distance y_{min} with finally reached orbital velocities $v_\phi(y_{min}) \sim 10^4$ km sec^{-1}, as shown in Table 4. The boundary inside which these NZRPs are trapped in circular orbits of radius y_{min} is shown by a small circle of radius $y = 0.05$. The compaction parameter is $u = 1.270 \times 10^{-4}$.

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